Statistical Physics

Introduction to Topological Order

Takashi Oka

1 Protection of a qubit

We ask how a many-body system can store quantum information and what makes this information vulnerable to local disturbances. Two small spin models will let us construct ground states, introduce stabilizers, and distinguish detectable errors from operations on an encoded qubit.

1.1 Conventions

The computational basis and Pauli matrices are

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus $X|0\rangle = |1\rangle$, $X|1\rangle = |0\rangle$, $Z|0\rangle = |0\rangle$, and $Z|1\rangle = -|1\rangle$. These matrices are dimensionless; a physical spin satisfies $S^\mu = (\hbar/2)\sigma^\mu$, where $(\sigma^x, \sigma^y, \sigma^z) = (X, Y, Z)$. The algebra is

$$X^2 = Y^2 = Z^2 = I, \qquad XY = -YX = iZ, \quad YZ = -ZY = iX, \quad ZX = -XZ = iY.$$

Equivalently,

$$[X, Y] = 2iZ, \qquad [Y, Z] = 2iX, \qquad [Z, X] = 2iY,$$

or $[\sigma^\mu, \sigma^\nu] = 2i \sum_\lambda \epsilon_{\mu\nu\lambda}\sigma^\lambda$. Operators on different sites commute; for example, $X_1 = X \otimes I$ and $Z_2 = I \otimes Z$ satisfy $[X_1, Z_2] = 0$.

The information in a qubit

A normalized pure state has the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

Its populations are $p_0 = |\alpha|^2$ and $p_1 = |\beta|^2$. When both amplitudes are nonzero, the relative phase is

$$\boxed{\phi = \arg\beta - \arg\alpha = \arg(\alpha^*\beta), \qquad e^{i\phi} = \frac{\alpha^*\beta}{|\alpha||\beta|}.}$$

An overall phase has no observable effect. The relative phase is undefined if either amplitude vanishes. In terms of measurable expectation values,

$$\langle Z\rangle = p_0 - p_1, \qquad \langle X\rangle + i\langle Y\rangle = 2\alpha^*\beta.$$

A quantum memory must preserve the relative phase as well as the populations.

1.2 Phase errors and bit flips for one qubit

A longitudinal field changes the relative phase

Let $H_{\text{eff}} = \epsilon Z$, where ϵ is a small energy scale. Then

$$|\psi(t)\rangle = \alpha e^{-i\epsilon t/\hbar}|0\rangle + \beta e^{i\epsilon t/\hbar}|1\rangle \sim \alpha|0\rangle + \beta e^{2i\epsilon t/\hbar}|1\rangle.$$

The symbol $\sim$ means equality up to an overall phase. The populations are unchanged, while

$$\phi \longmapsto \phi + \frac{2\epsilon t}{\hbar}.$$

An unknown field therefore introduces an unknown relative phase even when no population transfer occurs.

A transverse field mixes the basis states

For $H_{\text{eff}} = \epsilon X$, the identity $X^2 = I$ gives

$$e^{-i\epsilon X t/\hbar} = \cos\theta\, I - i\sin\theta\, X, \qquad \theta = \frac{\epsilon t}{\hbar}.$$

For example,

$$|0\rangle \longmapsto \cos\theta|0\rangle - i\sin\theta|1\rangle.$$

More generally, the amplitudes become

$$\alpha(t) = \alpha\cos\theta - i\beta\sin\theta, \qquad \beta(t) = \beta\cos\theta - i\alpha\sin\theta.$$

The populations generally change. At $\theta = \pi/2$, the evolution is $-iX$, which flips the computational basis states up to a common phase. At other times it is a coherent rotation, not necessarily a complete bit flip.

What does protection require?

Every Hermitian two-state Hamiltonian can be written as

$$H_{\text{eff}} = aI + b_x X + b_y Y + b_z Z.$$

The scalar term aI produces only a common phase. The other terms can change the stored state. To leave every state in the doublet unchanged up to an overall phase, the effective Hamiltonian must be proportional to the identity:

$$\boxed{H_{\text{eff}} = aI.}$$

After choosing an energy reference, this can be written as $H_{\text{eff}} = 0$. A known field can be used to perform a gate; an unknown or fluctuating field causes an unwanted operation. Unitary evolution under a fixed field alone does not turn a pure state into a mixed state.

An energy gap to higher states does not rule out an unwanted operation within a low-energy doublet.

1.3 Two spins and the Bell Hamiltonian

Consider

$$\boxed{H_{\text{Bell}} = -J(X_1X_2 + Z_1Z_2), \qquad J > 0.}$$

Write $K_1 = X_1X_2$ and $K_2 = Z_1Z_2$. The two local anticommutation signs cancel:

$$K_1K_2 = (X_1Z_1)(X_2Z_2) = (-Z_1X_1)(-Z_2X_2) = K_2K_1.$$

Thus $[K_1, K_2] = 0$, $[H_{\text{Bell}}, K_a] = 0$, and $K_a^2 = I$. We can choose a common eigenbasis with eigenvalues $k_1, k_2 = \pm 1$ and energies

$$E = -J(k_1 + k_2).$$

The lowest energy, $-2J$, requires $k_1 = k_2 = +1$. First, $ZZ|\psi\rangle = |\psi\rangle$ restricts the state to $c|00\rangle + d|11\rangle$. Applying XX then exchanges the two components, so $XX|\psi\rangle = |\psi\rangle$ requires $c = d$. After normalization,

$$\boxed{|\psi_{\text{gs}}\rangle = |\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad E_{\text{gs}} = -2J.}$$

The complete simultaneous eigenbasis consists of the four Bell states:

State	XX	ZZ	E_{Bell}/J	E_{F}/J
$\Psi^- = (\|01\rangle - \|10\rangle)/\sqrt{2}$	-1	-1	2	3
$\Phi^- = (\|00\rangle - \|11\rangle)/\sqrt{2}$	-1	$+1$	0	-1
$\Psi^+ = (\|01\rangle + \|10\rangle)/\sqrt{2}$	$+1$	-1	0	-1
$\Phi^+ = (\|00\rangle + \|11\rangle)/\sqrt{2}$	$+1$	$+1$	-2	-1

The Bell Hamiltonian has one ground state and an excitation gap $2J$. Its one-dimensional ground space does not itself store an arbitrary qubit.

Comparison with the ferromagnetic Heisenberg model

The last column gives the energies of

$$H_{\text{F}} = -J(XX + YY + ZZ).$$

It has the same simultaneous eigenbasis because

$$(XX)(ZZ) = -YY.$$

However, its ground space is the three-dimensional triplet, with energy $-J$; the singlet Ψ^- has energy $3J$. The condition $XX = ZZ = +1$ already fixes $YY = -1$, so adding $-JYY$ is not an additional compatible minimizing condition. The two-term Bell model provides a particularly simple example in which two independent conditions select a unique entangled state.

Polarization and correlations

For the Bell ground state,

$$\langle X_1\rangle = \langle Y_1\rangle = \langle Z_1\rangle = 0,$$

and the same holds for spin 2. Nevertheless,

$$\boxed{\langle X_1X_2\rangle = \langle Z_1Z_2\rangle = +1, \qquad \langle Y_1Y_2\rangle = -1.}$$

Individual spins have zero mean polarization, while pair measurements are perfectly correlated along X, Z and anticorrelated along Y.

Entanglement: what does one spin see?

The density matrix of the pair is

$$\begin{aligned}\rho_{12} &= |\Phi^+\rangle\langle\Phi^+| \\ &= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|).\end{aligned}$$

To trace out spin 2, use

$$\mathrm{Tr}_2\big(|a\rangle\langle a'|\otimes|b\rangle\langle b'|\big) = \langle b'|b\rangle|a\rangle\langle a'|.$$

For example, $\mathrm{Tr}_2|00\rangle\langle 11| = \langle 1|0\rangle|0\rangle\langle 1| = 0$. Both off-diagonal terms vanish under the partial trace, giving

$$\boxed{\rho_1 = \mathrm{Tr}_2\,\rho_{12} = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{I}{2}.}$$

The full state is pure, but either spin alone is maximally mixed. Its entanglement entropy is

$$S_1 = -\,\mathrm{Tr}\,\rho_1 \ln\rho_1 = -2\left(\frac{1}{2}\ln\frac{1}{2}\right) = \ln 2.$$

Natural logarithms are used here. For a pure bipartite state, this entropy measures entanglement. For a mixed global state it is not, by itself, an entanglement measure: the unentangled mixture $\rho_{\mathrm{mix}} = (|00\rangle\langle 00| + |11\rangle\langle 11|)/2$ also has $\rho_1 = I/2$, but has $\langle XX\rangle = 0$ rather than $+1$.

Stabilizers of a state

An operator K that satisfies $K|\psi\rangle = |\psi\rangle$ is said to stabilize that state. Here XX and ZZ are commuting Pauli stabilizers of Φ^+. Their simultaneous $+1$ eigenspace contains only this state up to scalar multiplication. Individually, each also has excited Bell states with eigenvalue $+1$. The word "stabilizer" refers to this fixed-state property; it does not by itself imply robustness against arbitrary disturbances.

From one state to a code space

Let $K_1, \ldots, K_r$ be independent commuting Hermitian Pauli operators on n qubits, with a nonempty common $+1$ eigenspace. Define

$$\boxed{\mathcal{C} = \{|\psi\rangle : K_a|\psi\rangle = |\psi\rangle \text{ for all } a\}.}$$

This is the *code space*. For such Pauli constraints, each independent condition halves the dimension:

$$\boxed{\dim \mathcal{C} = \frac{2^n}{2^r} = 2^{n-r}.}$$

Here r counts independent generators, not every product of them. For the Bell state, $K_1K_2 = -YY$ is already implied and does not add another constraint. Compatibility matters: the stabilizer group must not contain $-I$.

Frustration-free Hamiltonians

For

$$H = -\sum_{a=1}^{r} J_a K_a, \qquad J_a > 0,$$

every state in $\mathcal{C}$ minimizes each term separately. Thus $\mathcal{C}$ is the ground-state space and $E_0 = -\sum_a J_a$. This simultaneous minimization is the meaning of *frustration free*. Equivalently,

$$H - E_0 I = \sum_a J_a (I - K_a)$$

is a sum of positive semidefinite terms, each annihilating $\mathcal{C}$. Commutativity alone does not guarantee compatible minimizing conditions.

Projection onto the code space

Because $K_a^2 = I$, the projector onto its $+1$ eigenspace is

$$\boxed{P_a = \frac{I + K_a}{2}, \qquad P_a^2 = P_a.}$$

It has eigenvalues 1 and 0. The projectors commute, so their product is the projector onto all the conditions together:

$$P = \prod_{a=1}^{r} P_a.$$

For the Bell model,

$$P_{\text{Bell}} = \frac{(I + XX)(I + ZZ)}{4} = |\Phi^+\rangle\langle\Phi^+| = \frac{I + XX - YY + ZZ}{4}.$$

Thus $P_{\text{Bell}}|\chi\rangle = |\Phi^+\rangle\langle\Phi^+|\chi\rangle$. If the overlap is nonzero, normalizing the projected vector gives the ground state; a vector orthogonal to it is projected to zero.

Counting check. Expanding $P = 2^{-r}\prod_a (I + K_a)$, only the identity term has nonzero trace. Independence and compatibility exclude other products equal to $\pm I$. Therefore $\operatorname{Tr} P = 2^{n-r}$, the dimension of its image.

1.4 Three spins encode one qubit

Consider the open Ising chain at zero transverse field:

$$\boxed{H_0 = -J(Z_1Z_2 + Z_2Z_3), \qquad J > 0.}$$

The two stabilizers are $K_1 = Z_1Z_2$ and $K_2 = Z_2Z_3$. Their common $+1$ condition requires all three spins to agree, so

$$\mathcal{C} = \mathrm{span}\{|000\rangle, |111\rangle\}, \qquad \dim \mathcal{C} = 2^{3-2} = 2.$$

The projector is

$$P = \frac{(I + Z_1Z_2)(I + Z_2Z_3)}{4} = |000\rangle\langle 000| + |111\rangle\langle 111|.$$

Physical and logical qubits

The three spins are three *physical qubits*. We encode one *logical qubit* in their two-dimensional ground space, choosing

$$|\overline{0}\rangle = |000\rangle, \qquad |\overline{1}\rangle = |111\rangle, \qquad |\overline{\psi}\rangle = \alpha|\overline{0}\rangle + \beta|\overline{1}\rangle.$$

The bars label logical states. This encoding stores one qubit in a subspace; it does not create three copies of an unknown qubit. The quantum information includes the populations $|\alpha|^2, |\beta|^2$ and their relative phase.

The gap and domain walls

The energy for bond signs k_1, k_2 is $E = -J(k_1 + k_2)$:

Configurations	k_1	k_2	Energy
$000, 111$	$+1$	$+1$	$-2J$
$100, 011$	-1	$+1$	0
$001, 110$	$+1$	-1	0
$010, 101$	-1	-1	$2J$

The binary strings denote computational basis states. An antiparallel bond has $k_a = -1$ and is a *domain wall*. Each wall costs $2J$ relative to an aligned bond, and the gap above the code space is $2J$.

The gap imposes an energy cost on leaving the ground space. It does not guarantee preservation of the information *within* that space. We must test local disturbances and determine which ones can be detected.

Bit flips and error syndromes

A bit flip on the middle spin gives

$$X_2|\overline{\psi}\rangle = \alpha|010\rangle + \beta|101\rangle.$$

Since X_2 anticommutes with both checks,

$$K_1X_2|\overline{\psi}\rangle = -X_2|\overline{\psi}\rangle, \qquad K_2X_2|\overline{\psi}\rangle = -X_2|\overline{\psi}\rangle.$$

The measured signs (k_1, k_2) form the *error syndrome*:

Error	k_1	k_2	Correction
I	$+1$	$+1$	I
X_1	-1	$+1$	X_1
X_2	-1	-1	X_2
X_3	$+1$	-1	X_3

Assumption: at most one physical bit flip. Under this assumption the syndrome identifies a correction, for example $X_2X_2|\overline{\psi}\rangle = |\overline{\psi}\rangle$.

Both 010 and 101 have syndrome $(-1, -1)$, independently of α, β. An ideal measurement of the checks therefore preserves their superposition. Measuring Z_1, Z_2, Z_3 individually would reveal more information and destroy this coherence. A check result $+1$ means that the corresponding constraint is satisfied; it does not certify that no error occurred.

Undetected phase changes and logical operations

A local phase flip gives

$$Z_1|\overline{\psi}\rangle = \alpha|000\rangle - \beta|111\rangle.$$

It preserves both checks. The syndrome remains $(+1, +1)$ even though the logical state generally changes. Flipping all three spins also preserves the checks:

$$X_1X_2X_3|\overline{\psi}\rangle = \alpha|111\rangle + \beta|000\rangle.$$

On the code space, define the logical Pauli operators

$$\boxed{\overline{Z} = Z_1\big|_{\mathcal{C}}, \qquad \overline{X} = (X_1X_2X_3)\big|_{\mathcal{C}}.}$$

They satisfy $\overline{XZ} = -\overline{ZX}$. A stabilizer acts as the identity on $\mathcal{C}$; a nontrivial logical operator preserves $\mathcal{C}$ but can change the encoded information. An intended logical operation is a gate; an unintended one is a logical error.

The logical X has support on all three spins. Under independent bit-flip noise it requires multiple errors, whereas a logical phase flip needs only one Z_j. This repetition code corrects single bit flips, not arbitrary single-qubit errors. Multiple bit flips can also imitate a single-error syndrome and lead to an incorrect recovery.

1.5 The local order parameter is the weakness

Add a weak local field $V = \epsilon Z_1$ to H_0, with $|\epsilon| \ll J$. Within the ordered basis $(|000\rangle, |111\rangle)$,

$$(H_0 + V)\big|_{\mathcal{C}} = -2JI_2 + \epsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The splitting is $2|\epsilon|$. Up to the common phase from $-2J$,

$$|\overline{\psi}(t)\rangle \sim \alpha e^{-i\epsilon t/\hbar}|\overline{0}\rangle + \beta e^{i\epsilon t/\hbar}|\overline{1}\rangle.$$

The relative phase changes without creating a domain wall or leaving the ground space of H_0. Both check outcomes remain $+1$.

Why does a local field distinguish the states?

The same operator is the local order-parameter operator of the Ising model:

$$\langle 000|Z_1|000\rangle = +1, \qquad \langle 111|Z_1|111\rangle = -1.$$

Its different expectation values make the two configurations locally distinguishable and let a field coupled to Z_1 shift their energies differently. This field breaks the global spin-flip symmetry $Q = X_1X_2X_3$, since $QZ_1Q = -Z_1$.

A degeneracy distinguishable by a local order parameter can be split by a local field coupled to that order parameter.

A criterion for the action of a local perturbation

For a Hermitian local operator O, the first-order perturbation within the code space is POP. In a logical basis, it is a 2×2 matrix:

$$POP\big|_{\mathcal{C}} = \begin{pmatrix} \langle\overline{0}|O|\overline{0}\rangle & \langle\overline{0}|O|\overline{1}\rangle \\ \langle\overline{1}|O|\overline{0}\rangle & \langle\overline{1}|O|\overline{1}\rangle \end{pmatrix}.$$

Unequal diagonal entries distinguish the logical states; off-diagonal entries mix them. A scalar action avoids both effects at first order:

$$\boxed{POP = c_O P, \qquad \text{equivalently} \qquad POP\big|_{\mathcal{C}} = c_O I_{\mathcal{C}}.}$$

The repetition code fails this condition for $O = Z_1$. By contrast, $PX_jP = 0$: one bit flip takes the state to a subspace orthogonal to $\mathcal{C}$. But $P(X_1X_2X_3)P$ acts as $\overline{X}$ within $\mathcal{C}$.

The scalar-action condition must hold for every perturbation in the specified allowed class. It tests first-order splitting within the subspace; by itself it neither excludes higher-order virtual processes or leakage nor establishes complete quantum error correction.