Statistical Physics

Introduction to Topological Order

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2 Quantum Ising model

Spins are fixed on a lattice, but their excitations can move like particles. We study the two limits of the Ising chain, use domain-wall motion to explain tunneling between ordered states, and finally connect to Lecture 1.

2.1 The model and its symmetry

For an open chain of N spin-1/2 degrees of freedom,

$$\boxed{H = -J\sum_{i=1}^{N-1} Z_iZ_{i+1} - h\sum_{i=1}^{N} X_i, \qquad J>0, \quad h\geq 0.}$$

Here X, Y, Z are Pauli matrices. We use

$$|0\rangle = |\uparrow\rangle, \quad |1\rangle = |\downarrow\rangle, \qquad |\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}, \quad X|\pm\rangle = \pm|\pm\rangle.$$

The exchange J favors alignment along Z, whereas h favors alignment along X. Their competition produces quantum fluctuations.

Recall the classical Ising model and the Landau picture

At $h = 0$, a configuration has $Z_i = s_i = \pm 1$. Its energy and the local magnetization order parameter are

$$E = -J\sum_{i=1}^{N-1} s_is_{i+1}, \qquad m = \frac{1}{N}\sum_i \langle Z_i\rangle.$$

The transformation $s_i \mapsto -s_i$ at every site leaves E unchanged. In a Landau description of an Ising transition, symmetry requires an even free-energy density: $f(m) = f_0 + \frac{1}{2}am^2 + \frac{1}{4}bm^4 + \cdots$, with $b > 0$. Its minimum is at $m = 0$ for $a > 0$ and at $m = \pm\sqrt{-a/b}$ for $a < 0$. Choosing one of the two signs breaks the symmetry.

One-dimensional qualification. This is a general Landau picture, not an exact finite-temperature description of this chain. The classical one-dimensional nearest-neighbor Ising model has no ordered phase at any $T > 0$: $T_c = 0$. From now on we study the quantum model at $T = 0$ and vary h/J.

The quantum symmetry. Define the global spin flip

$$Q = \prod_{i=1}^{N} X_i, \qquad Q^2 = I.$$

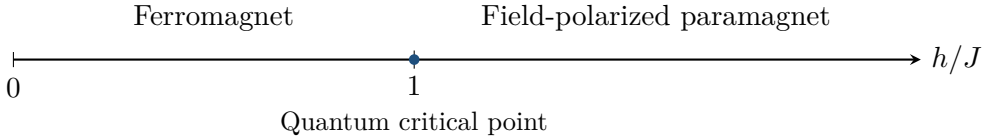
Since $QZ_iQ = -Z_i$ and $QX_iQ = X_i$, every bond is unchanged: $\boxed{[H, Q] = 0}$. This is the $\mathbb{Z}_2$ Ising symmetry. A longitudinal field $-g\sum_i Z_i$ breaks it when $g \neq 0$; the transverse field preserves it.

2.2 Strong field: independent spins

Two limits and emergent particles

Interactions among many microscopic degrees of freedom can produce phases, spontaneous symmetry breaking, and propagating quasiparticles. These are examples of *emergence*, often associated with Anderson's phrase "More Is Different." In this model the Ising symmetry is already present in the Hamiltonian; the collective phenomenon is its spontaneous breaking.

The zero-temperature phase diagram of the uniform infinite chain is



The exact boundary $h = J$ will follow from the solution in Lecture 3. Here we build the physical picture near the two endpoints. The product states below are exact only at those endpoints, not throughout either phase.

The ground state at $J = 0$

For $h > 0$,

$$H_0 = -h \sum_i X_i, \qquad \boxed{|\Omega_x\rangle = |+\rangle^{\otimes N}, \qquad E_0 = -Nh.}$$

Each spin independently minimizes its field energy. This unique ground state is a product state with no entanglement between sites. It preserves the symmetry:

$$Q|\Omega_x\rangle = |\Omega_x\rangle, \qquad \langle Z_i \rangle = 0, \qquad \langle Z_i Z_j \rangle = 0 \quad (i \neq j).$$

The on-site value is instead $\langle Z_i^2 \rangle = 1$.

A reversed spin is an excitation

The operator Z_j reverses the X spin because $Z|+\rangle = |-\rangle$:

$$|j\rangle = Z_j|\Omega_x\rangle = |+\cdots+\ -_j\ +\cdots+\rangle.$$

Its energy is $E_0 + 2h$. For distinct j, k, the state $|j,k\rangle = Z_j Z_k|\Omega_x\rangle$ has energy $E_0 + 4h$. More generally,

$$r \text{ reversed spins:} \qquad E = E_0 + 2hr.$$

The one-spin excitation is called a *spin-flip quasiparticle*, or a *magnon* in this field-polarized setting. At $J = 0$ it does not move; all N possible positions have the same energy.

Here a spin flip means a reversal in the X basis, produced by Z_j. The bit flip X_j of Lecture 1 reverses the computational (Z) basis. Always specify the background and the basis.

2.3 A spin flip can move

Turn on a small exchange, $J/h \ll 1$. All one-flip states $|j\rangle$ are degenerate at $J = 0$, so we first diagonalize the perturbation within their span. Let $P_1 = \sum_j |j\rangle\langle j|$ project onto this subspace.

The hopping matrix element

On a bond touching the reversed spin,

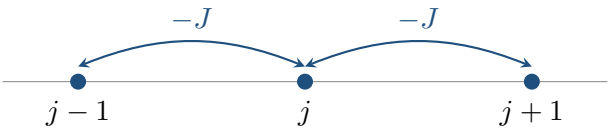
$$Z_j Z_{j+1} |\cdots -_j +_{j+1} \cdots\rangle = |\cdots +_j -_{j+1} \cdots\rangle.$$

Thus the exchange moves the excitation by one site with amplitude $-J$. A bond away from the excitation creates two additional flips. Those three-flip states are outside the subspace and do not contribute at first order. For an interior site,

$$P_1 \left(\sum_i Z_i Z_{i+1} \right) |j\rangle = |j-1\rangle + |j+1\rangle.$$

The matrix elements of the first-order projected Hamiltonian are therefore

$$\langle j|H|k\rangle = (E_0 + 2h)\delta_{jk} - J(\delta_{j,k+1} + \delta_{j,k-1}), \qquad E_0 = -Nh.$$



From hopping to a dispersion relation

For $|\psi\rangle = \sum_j \psi_j |j\rangle$, write the excitation energy as ϵ:

$$\boxed{\epsilon\psi_j = 2h\psi_j - J(\psi_{j-1} + \psi_{j+1}).}$$

To calculate the bulk dispersion, temporarily use a periodic chain and set the lattice spacing to one. Substituting

$$\psi_j = \frac{e^{ikj}}{\sqrt{N}}, \qquad k = \frac{2\pi n}{N},$$

gives

$$\boxed{\epsilon_{\text{flip}}(k) = 2h - 2J\cos k + O(J^2/h).}$$

The ground-state energy has no first-order correction in J; virtual multi-flip states contribute at higher orders. The minimum of this band is

$$\Delta_{\text{flip}} = 2h - 2J + O(J^2/h).$$

For a finite open chain, the same first-order hopping problem has standing waves $\psi_j \propto \sin(kj)$ with $k = \pi m/(N+1)$, $m = 1, \ldots, N$. Its bulk band is the same.

The spins remain on their sites; the reversed-spin excitation travels. This is a simple example of a propagating quasiparticle.

The expansion is controlled for $J/h \ll 1$. Extrapolating its gap to zero suggests a transition, but does not prove that its exact location is $h = J$. That conclusion requires the exact solution.

2.4 Weak field: order and domain walls

Return to the open chain and set $h = 0$. The two ground states are

$$|U\rangle = |00\cdots0\rangle, \qquad |D\rangle = |11\cdots1\rangle, \qquad E_0 = -J(N-1).$$

They have opposite local magnetizations, $\langle Z_i\rangle = +1$ and -1, and Q exchanges them.

The energy counts misaligned bonds

For a Z-basis configuration, define

$$n_{i+1/2} = \frac{1 - s_i s_{i+1}}{2} = 0 \text{ or } 1.$$

A misaligned bond is a *domain wall* (DW), separating regions with opposite magnetizations. Its energy cost is $2J$:

$$\boxed{E = E_0 + 2J \sum_{i=1}^{N-1} n_{i+1/2}.}$$

As an operator, $n_{i+1/2} = (I - Z_i Z_{i+1})/2$. For example,

$$|0000000\rangle \longrightarrow |0011100\rangle$$

creates two walls and costs $4J$. Enlarging the reversed interior block does not change the cost: the energy resides at its two boundaries.

A string creates a wall at its endpoint

For $1 \le a < N$, define

$$W_a = X_1 X_2 \cdots X_a, \qquad W_a|U\rangle = |\underbrace{11\cdots1}_{a}00\cdots0\rangle.$$

Only the bond $(a, a+1)$ is misaligned, so the energy cost is $2J$. Every bond inside the reversed segment has both spins flipped and remains aligned. When the string reaches the far end,

$$\boxed{W_N = Q, \qquad W_N|U\rangle = |D\rangle, \qquad \Delta E = 0.}$$

There is then no wall left. A single wall in the bulk of an infinite chain requires a half-infinite string; a finite interior string has two endpoints.

Open and periodic boundaries

On a ring, $Z_{N+1} = Z_1$ and $\prod_{i=1}^{N} Z_i Z_{i+1} = I$. Consequently, the number of walls is even. For $N \ge 3$,

At $h = 0$	Open chain	Periodic chain
Ground-state energy	$-J(N-1)$	$-JN$
Ground states	$\|U\rangle, \|D\rangle$	$\|U\rangle, \|D\rangle$
Allowed wall number	$0, 1, 2, \ldots$	$0, 2, 4, \ldots$
Lowest excitation cost	$2J$	$4J$

A domain wall is a particle-like defect on a bond. Its creation energy depends on J, while the transverse field will provide its motion.

2.5 Domain-wall motion

Now let $h/J \ll 1$. On an open chain, denote a one-wall configuration by

$$|a\rangle_{\mathrm{DW}} = W_a|U\rangle = |1\cdots 1_a\, 0_{a+1}\cdots 0\rangle, \qquad 1 \le a < N.$$

For a wall away from the ends, flipping either adjacent spin moves it:

$$X_a|a\rangle_{\mathrm{DW}} = |a-1\rangle_{\mathrm{DW}}, \qquad X_{a+1}|a\rangle_{\mathrm{DW}} = |a+1\rangle_{\mathrm{DW}}.$$

$|1\,1|0\,0\,0\,0\rangle$

$\updownarrow\ -hX_a$

$|1\,1\,1|0\,0\,0\rangle$ The blue mark is the wall.

$\updownarrow\ -hX_{a+1}$

$|1\,1\,1\,1|0\,0\rangle$

The hopping matrix element is $-h$. In the bulk, a spin flip away from the wall creates two extra walls and leaves the one-wall subspace. Projecting onto that subspace gives the first-order equation

$$\epsilon\psi_a = 2J\psi_a - h(\psi_{a-1} + \psi_{a+1}).$$

To describe bulk propagation, take an infinite chain with opposite magnetizations far to the left and right. A plane wave then gives

$$\psi_a \propto e^{ik(a+1/2)}, \qquad \boxed{\epsilon_{\mathrm{DW}}(k) = 2J - 2h\cos k + O(h^2/J).}$$

The coordinate $a + 1/2$ locates the wall on a bond. On a finite open chain, the first-order projected problem instead has standing waves. At an end, a spin flip can annihilate the wall into $|U\rangle$ or $|D\rangle$; those states are outside the one-wall subspace. On a ring, walls occur in pairs, so the dispersion does not imply that a single-wall state is allowed there.

Compare the two particles

	Spin flip, $J/h \ll 1$	Domain wall, $h/J \ll 1$
Position	Site j	Bond $a + 1/2$
Creation cost at the limit	$2h$	$2J$
Hopping matrix element	$-J$	$-h$
First-order dispersion	$2h - 2J\cos k$	$2J - 2h\cos k$

The exchange of J and h foreshadows duality. Both dispersions will be recovered from the exact solution in Lecture 3.

The background determines what moves: a reversed spin in the field-polarized phase, or a boundary between ordered regions in the ferromagnet.

2.6 Finite chains: tunneling and the split doublet

For a finite open chain, a nonzero transverse field lifts the exact $h = 0$ ground-state degeneracy. In the weak-field regime the two lowest states approach

$$|C_\pm\rangle = \frac{|U\rangle \pm |D\rangle}{\sqrt{2}}, \qquad Q|C_\pm\rangle = \pm|C_\pm\rangle.$$

They acquire different energies even though $[H, Q] = 0$: an Abelian symmetry does not require its two sectors to be degenerate. At finite h, the exact states also contain configurations with domain walls; the expressions above are their $h \to 0$ limits at fixed N.

In the low-energy effective basis associated with $|U\rangle, |D\rangle$, write

$$H_{\text{eff}} = E_c I + \begin{pmatrix} 0 & -t_N \\ -t_N & 0 \end{pmatrix}, \qquad t_N > 0.$$

The symmetric state is lower, with energies $E_c - t_N$ and $E_c + t_N$. Thus the doublet splitting is

$$\boxed{\delta E = 2t_N.}$$

The common energy E_c includes perturbative shifts of both states. The scale $t_N \sim J(h/J)^N$ is very small in a long, weak-field chain. What physical process produces it?

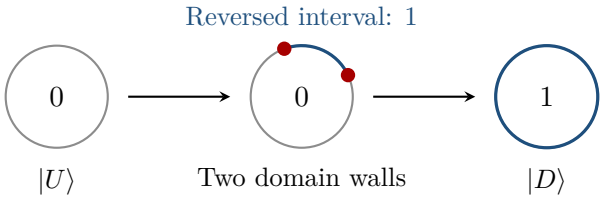
Create a wall, move it, and remove it

Starting from the left end, successive applications of X_i give

$$|000\cdots0\rangle \xrightarrow{X_1} |100\cdots0\rangle \xrightarrow{X_2} |110\cdots0\rangle \longrightarrow |11\cdots10\rangle \xrightarrow{X_N} |11\cdots1\rangle.$$

At $h = 0$ every intermediate configuration in this particular path has one wall and costs $2J$. The first flip creates it, the intermediate flips move it, and the last flip removes it. The process connects the two aligned states.

On a ring, a local flip creates two walls. Growing the reversed interval separates the walls until they meet and annihilate, leaving the opposite magnetization:



In the previous section, a wall was an actual excitation. Here walls appear as virtual intermediate states in a low-energy tunneling process; they need not be thermally created.

The tunneling matrix element in perturbation theory

Split the Hamiltonian into

$$H_0 = -J\sum_{i=1}^{N-1} Z_i Z_{i+1}, \qquad V = -h\sum_i X_i, \qquad E_0 = -J(N-1).$$

To connect $|U\rangle$ and $|D\rangle$, every spin must be flipped an odd number of times. The fewest possible applications of V is therefore N. All lower-order off-diagonal matrix elements vanish. The leading term is

$$A_N = \langle D|H_{\text{eff}}^{(N)}|U\rangle = {\sum_{n_1,\ldots,n_{N-1}}}' \frac{\langle D|V|n_{N-1}\rangle\langle n_{N-1}|V|n_{N-2}\rangle\cdots\langle n_1|V|U\rangle}{(E_0-E_{n_{N-1}})\cdots(E_0-E_{n_1})}.$$

The prime excludes $|U\rangle$ and $|D\rangle$ from the intermediate states. Each numerator factor flips one spin and contributes $-h$; each denominator is minus the energy cost of the intermediate walls. This expression has units of energy: it is a Hamiltonian matrix element, not a coefficient in a wavefunction expansion.

One path versus the sum over paths

The path that flips sites in the order $1, 2, \ldots, N$ has one wall at every intermediate step, giving

$$A_N^{\text{one path}} = \frac{(-h)^N}{(-2J)^{N-1}} = -\frac{h^N}{(2J)^{N-1}}.$$

Other flip orders can contain several walls. They must also be included; one end-to-end path alone does not give the full coefficient.

For a two-site open chain, the two paths go through $|10\rangle$ or $|01\rangle$, both at energy $E_0 + 2J$. Hence

$$A_2 = \frac{(-h)^2}{-2J} + \frac{(-h)^2}{-2J} = -\frac{h^2}{J}, \qquad \delta E = 2\frac{h^2}{J} + O(h^4/J^3).$$

For the uniform open chain, the complete leading-order sum gives

$$A_N = -\frac{h^N}{J^{N-1}}, \qquad t_N \simeq -A_N,$$

and therefore, at fixed $N \geq 2$ as $h/J \to 0$,

$$\boxed{\delta E = 2J\left(\frac{h}{J}\right)^N + O\left(\frac{h^{N+2}}{J^{N+1}}\right).}$$

The general path sum is stated here without evaluating every ordering. The physical reason for the high order is simple: a process must traverse the whole chain to reverse every spin. The periodic-chain denominators and prefactor differ, because its intermediate configurations contain an even number of walls.

Two different energy scales

In the ordered phase, the doublet splitting becomes exponentially small with chain length, while the bulk excitation gap stays finite away from the transition. In the weak-field limit these scales are

$$\delta E \sim J(h/J)^N, \qquad \Delta_{\text{DW}} = 2J - 2h + O(h^2/J).$$

The perturbative formula above is a fixed-N, small-h expansion. The exact solution in Lectures 3–4 establishes the exponential splitting at fixed $0 < h/J < 1$ as N grows.

A small finite-size splitting and a finite bulk gap can coexist. Restoring the ordered doublet as $N \to \infty$ does not make the bulk gapless.

2.7 The quantum-information dictionary

We can now reinterpret the spin physics using Lecture 1. At $h = 0$,

$$\mathcal{C} = \mathrm{span}\{|U\rangle, |D\rangle\}, \qquad |0_L\rangle = |U\rangle, \quad |1_L\rangle = |D\rangle$$

is the repetition-code space, with stabilizers $K_i = Z_i Z_{i+1}$.

Ising description	Quantum-information description
Aligned bond, $Z_i Z_{i+1} = +1$	Satisfied stabilizer check
Domain wall, $Z_i Z_{i+1} = -1$	Check violation in the syndrome
Two aligned ground states	Logical basis $\|0_L\rangle, \|1_L\rangle$
Full spin flip $Q = \prod_i X_i$	Logical $\overline{X}$ on $\mathcal{C}$
Local order parameter Z_j	Logical $\overline{Z}$ on $\mathcal{C}$

A full DW traversal implements a logical bit flip: $W_N = \overline{X}$ on $\mathcal{C}$. Virtual tunneling generates the effective term $-t_N \overline{X}$. At $h = 0$, a partial string leaves a detectable wall; the full string leaves none.

What does a syndrome measurement reveal?

For $1 \le a < N$, a partial-string error produces

$$W_a(\alpha|U\rangle + \beta|D\rangle) = \alpha W_a|U\rangle + \beta W_a|D\rangle.$$

Both components have the same violated bond $(a, a+1)$. Measuring the checks therefore identifies the wall without distinguishing the two logical components or revealing their amplitudes. A syndrome is the pattern of check outcomes, not a unique identification of the error.

The local order parameter is also the weakness

On $\mathcal{C}$, $Z_1 = \overline{Z}$. A local field $-gZ_1$ gives

$$\alpha|U\rangle + \beta|D\rangle \longmapsto \alpha e^{igt/\hbar}|U\rangle + \beta e^{-igt/\hbar}|D\rangle.$$

It changes the relative phase without creating any wall. At $h = 0$ it splits the two states by $2|g|$, independent of N. Thus suppressing the transverse tunneling amplitude does not protect against local phase noise.

The same local operator that distinguishes the Ising magnetizations acts as a logical phase operator. This code protects against certain bit-flip errors, but does not protect an arbitrary qubit against general local noise.

Scope. The ground space is exactly this stabilizer code at $h = 0$. For $h \neq 0$, the checks do not commute with H; the low-energy states contain quantum fluctuations and are not simultaneous $+1$ eigenstates of all K_i. In the ordered phase, spontaneous magnetization is selected by taking $N \to \infty$ in a small longitudinal field before removing that field.

Next. The Jordan–Wigner solution will recover both quasiparticle dispersions and explain the small doublet splitting. It will also show why the allowed local perturbations differ between a spin chain and a physical superconducting wire.

Further reading. L. Balents, *Notes for Physics 217A* (2015), Sec. 2; A. Kitaev and C. Laumann, *Topological phases and quantum computation*, Sec. 2; P. W. Anderson, *More Is Different*, Science **177**, 393–396 (1972).