Statistical Physics

Introduction to Topological Order

Takashi Oka

3 The Kitaev chain and the Ising model

Lecture 2 described the Ising chain using spin flips and domain walls. Today we start with a superconducting fermion chain, solve its bulk spectrum, and distinguish two phases by drawing a circle. The Jordan–Wigner transformation will then identify this solvable model with the Ising chain.

3.1 A superconducting chain of spinless fermions

Each site has one orbital, either empty or occupied. The operators obey

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \qquad \{c_i, c_j\} = 0, \qquad n_j = c_j^\dagger c_j, \quad n_j^2 = n_j.$$

The Kitaev Hamiltonian on an open chain is

$$\begin{aligned} H_{\mathrm{K}} = & -w \sum_{j=1}^{N-1} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu \sum_{j=1}^{N} (n_j - \tfrac{1}{2}) \\ & + \sum_{j=1}^{N-1} \left(\Delta_{\mathrm{p}} c_j c_{j+1} + \Delta_{\mathrm{p}}^* c_{j+1}^\dagger c_j^\dagger \right). \end{aligned}$$

- $w > 0$ is the hopping amplitude: a fermion moves between neighbors.
- μ is the chemical potential: it controls the cost of occupation.
- The Δ_{p} term removes a pair on neighboring sites; its Hermitian conjugate creates a pair, with the reversed operator order shown above.

There is only one fermion species. Its pair amplitude is odd under exchange, the one-dimensional version of p-wave pairing. We use a fixed, uniform condensate phase and take Δ_{p} real. This is an effective superconducting Hamiltonian; we do not derive the condensate itself.

Number changes by two; parity is conserved

Let $N_f = \sum_j n_j$ and $Q_f = (-1)^{N_f}$. Pair creation changes N_f by two:

$$[H_{\mathrm{K}}, N_f] \neq 0 \quad (\Delta_{\mathrm{p}} \neq 0), \qquad \boxed{[H_{\mathrm{K}}, Q_f] = 0.}$$

The wire can exchange pairs with a condensate while preserving fermion parity. The pairing amplitude Δ_{p} is *not* the quasiparticle gap; that gap must be calculated.

Our main example has $\Delta_{\mathrm{p}} = w > 0$. This makes the geometry a circle and will give the Ising model exactly. We first keep Δ_{p} explicit to see where the terms in the dispersion come from.

We set the lattice spacing and $\hbar$ to one. The next calculation uses an infinite chain (or a fermionic ring) to find the bulk spectrum. The later Jordan–Wigner mapping is established on an open chain.

3.2 Fourier transformation: why momenta come in pairs

Use $c_j = N^{-1/2}\sum_k e^{ikj}c_k$, with $-\pi < k \le \pi$. The hopping and chemical-potential terms give $\xi_k = -\mu - 2w\cos k$. For the pairing term,

$$\sum_j c_j c_{j+1} = \sum_k e^{-ik} c_k c_{-k} = -2i \sum_{0<k<\pi} \sin k\, c_k c_{-k}.$$

The last equality uses $c_{-k}c_k = -c_k c_{-k}$. Thus

$$\boxed{\xi_k = -\mu - 2w\cos k, \qquad \delta_k = 2\Delta_{\mathrm{p}} \sin k.}$$

Pairing couples k to $-k$ and is odd in momentum. The special momenta $0, \pi$, when allowed by the boundary condition, have $\delta_k = 0$.

3.3 The BdG matrix and the quasiparticle dispersion

We use the Bogoliubov–de Gennes (BdG) formalism from undergraduate superconductivity. With the Nambu column $\Psi_k = (c_k, c_{-k}^\dagger)^T$,

$$H_{\mathrm{K}} = \frac{1}{2}\sum_k \Psi_k^\dagger \mathcal{H}(k) \Psi_k, \qquad \boxed{\mathcal{H}(k) = \begin{pmatrix} \xi_k & -i\delta_k \\ i\delta_k & -\xi_k \end{pmatrix} = \xi_k \tau_z + \delta_k \tau_y.}$$

Here τ acts in particle–hole space, not on the original physical spins. The sum is over the full Brillouin zone; $1/2$ removes the Nambu double counting. This expression includes the constant from $-\mu(n_j - 1/2)$.

Since $\tau_z^2 = \tau_y^2 = I$ and $\{\tau_z, \tau_y\} = 0$,

$$\mathcal{H}(k)^2 = (\xi_k^2 + \delta_k^2) I.$$

The BdG eigenvalues are $\pm E(k)$, with

$$\boxed{E(k) = \sqrt{(\mu + 2w\cos k)^2 + 4\Delta_{\mathrm{p}}^2 \sin^2 k}.}$$

A Bogoliubov transformation gives

$$H_{\mathrm{K}} = E_0 + \sum_k E(k) \eta_k^\dagger \eta_k.$$

The η_k mix c_k and $c_{-k}^\dagger$.

Creating one quasiparticle costs $E(k)$. The BdG eigenvalues $\pm E(k)$ encode creation and annihilation, not two independent excitations of opposite energy. At fixed total parity, quasiparticle occupations must respect that constraint.

The quasiparticle vacuum is a paired state

For a distinct pair $(k, -k)$, write

$$|\Omega_k\rangle = u_k|00\rangle + v_k|11\rangle, \qquad |11\rangle = c_k^\dagger c_{-k}^\dagger |00\rangle, \quad |u_k|^2 + |v_k|^2 = 1.$$

For example, at $\xi_k = 0$ and $\delta_k > 0$,

$$|\Omega_k\rangle = \frac{|00\rangle + i|11\rangle}{\sqrt{2}}, \qquad \eta_k = \frac{c_k - i c_{-k}^\dagger}{\sqrt{2}}, \quad \eta_{-k} = \frac{c_{-k} + i c_k^\dagger}{\sqrt{2}}.$$

Both η_k and η_{-k} annihilate this state: it is empty of quasiparticles, but not of physical fermions. Pairing fixes a relative phase between empty and occupied pairs. The eigenvectors therefore carry information beyond $E(k)$; the circle below uses this information. Supplement A gives a direct two-state check and the general coefficients.

3.4 Two phases seen as a circle in the complex plane

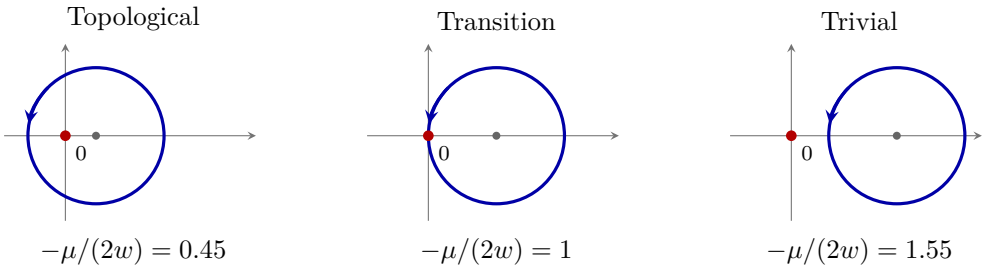
The BdG matrix contains both ξ_k and δ_k. Combine them into

$$q(k) = \xi_k - i\delta_k = -\mu - 2w\cos k - 2i\Delta_\mathrm{p}\sin k, \qquad E(k) = |q(k)|.$$

For $\Delta_\mathrm{p} = w$, this becomes

$$\boxed{q(k) = -\mu - 2we^{ik}.}$$

As k runs through the Brillouin zone, $q(k)$ traces a circle centered at $-\mu$ on the real axis, with radius $2w$. The distance from the origin is the quasiparticle energy. The shortest distance is the bulk gap.



Circles for $q/(2w)$. Horizontal and vertical axes are its real and imaginary parts. The red point is the origin.

Condition	Winding around zero	Bulk gap, $\Delta_\mathrm{p} = w$
$\lvert\mu\rvert < 2w$	Once	$2w - \lvert\mu\rvert$
$\lvert\mu\rvert > 2w$	None	$\lvert\mu\rvert - 2w$
$\lvert\mu\rvert = 2w$	Curve passes through zero	0

The winding cannot change continuously without $q(k) = 0$ at some momentum, which closes the gap. For this real-coefficient model the winding is $\nu = (2\pi)^{-1}\int_{-\pi}^{\pi} dk\, \partial_k \arg q(k)$, equal to 1 or 0 above. The general fermionic topological distinction retains its parity, $\mathcal{M} = (-1)^\nu$: topological -1, trivial $+1$.

What information is topological? $E(k)$ gives only the length of $q(k)$. Its direction also enters the eigenvectors of the BdG matrix. We use this matrix information, not the dispersion alone, to define winding. The real-space sites and Fourier basis are fixed throughout this comparison.

For real $\Delta_\mathrm{p} > 0$ different from w, the circle becomes an ellipse. It still encloses zero precisely when $|\mu| < 2w$. Thus the phase occupies a region, not just the solvable line $\Delta_\mathrm{p} = w$.

The edge interpretation of $\mathcal{M} = -1$ will be demonstrated in Lecture 4. The circle argument identifies the bulk distinction; it does not yet construct a boundary mode.

3.5 Jordan–Wigner: a spin chain hidden inside the fermions

We now return to the spins of Lecture 2. A spin has two states, just as a fermion orbital has two occupations. In the transverse-field basis,

$$X|\pm\rangle = \pm|\pm\rangle, \qquad |+\rangle \longleftrightarrow n = 0, \qquad |-\rangle \longleftrightarrow n = 1.$$

The local operator removing an occupation is

$$s_j = |+\rangle_j\langle -|_j = \frac{Z_j - iY_j}{2}, \qquad s_j^\dagger s_j = \frac{1 - X_j}{2}.$$

On one site $s_j^2 = 0$ and $\{s_j, s_j^\dagger\} = I$. On different sites, however, s_i, s_j commute, while c_i, c_j must anticommute.

The string supplies the exchange sign

Use the partial spin-flip string of Lecture 2, $W_a = \prod_{\ell=1}^{a} X_\ell$:

$$W_{j-1} = \prod_{\ell<j} X_\ell, \qquad \boxed{c_j = W_{j-1}s_j, \qquad c_j^\dagger = W_{j-1}s_j^\dagger.}$$

The string stops just before site j; the empty product is $W_0 = I$. Since $W_{j-1}^2 = I$ and $[W_{j-1}, s_j] = 0$,

$$n_j = \frac{1 - X_j}{2}, \qquad W_{j-1} = (-1)^{\sum_{\ell<j} n_\ell}.$$

For $i < j$, the string in c_j contains X_i, which anticommutes with s_i and $s_i^\dagger$. This gives exactly one minus sign; all other factors commute. Therefore

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \qquad \{c_i, c_j\} = 0.$$

Two-site check. Use the ordered basis $|11\rangle = c_1^\dagger c_2^\dagger|00\rangle$. Then

$$c_2|11\rangle = -|10\rangle, \quad c_1|11\rangle = |01\rangle, \qquad c_1c_2|11\rangle = -|00\rangle, \quad c_2c_1|11\rangle = |00\rangle.$$

The sign is the parity of the occupations to the left of the removed fermion. In general,

$$c_j|n_1 \cdots n_N\rangle = (-1)^{\sum_{\ell<j} n_\ell} n_j|n_1 \cdots 0_j \cdots n_N\rangle.$$

The map is invertible on the open chain:

$$X_j = 1 - 2n_j, \qquad Z_j = W_{j-1}(c_j + c_j^\dagger).$$

Both descriptions have 2^N states. The Ising spin-flip symmetry becomes fermion parity:

$$\boxed{Q = \prod_j X_j = (-1)^{\sum_j n_j} = Q_f.}$$

The two-state identification fixes occupation. The Jordan–Wigner string fixes statistics. Both ingredients are needed for an operator mapping.

3.6 One bond proves the correspondence

Set $\Delta_{\mathrm{p}} = w$. The hopping and pairing on one bond combine into

$$\begin{aligned} -w(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + w(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger) \\ = w(c_j - c_j^\dagger)(c_{j+1} + c_{j+1}^\dagger). \end{aligned}$$

The Jordan–Wigner formulas imply

$$c_j - c_j^\dagger = -iW_{j-1}Y_j, \qquad c_{j+1} + c_{j+1}^\dagger = W_j Z_{j+1}, \qquad W_j = W_{j-1}X_j.$$

Using $YX = -iZ$, all the strings cancel:

$$\begin{aligned} (c_j - c_j^\dagger)(c_{j+1} + c_{j+1}^\dagger) &= -iY_j X_j Z_{j+1} \\ &= -Z_j Z_{j+1}. \end{aligned}$$

The chemical-potential term is equally simple:

$$-\mu(n_j - \tfrac{1}{2}) = \frac{\mu}{2} X_j.$$

Hence the open superconducting chain is exactly

$$H_{\mathrm{K}} = -w \sum_{j=1}^{N-1} Z_j Z_{j+1} + \frac{\mu}{2} \sum_{j=1}^{N} X_j.$$

Comparing with $H_{\mathrm{Ising}} = -J \sum_j Z_j Z_{j+1} - h \sum_j X_j$ gives

$$\boxed{w = \Delta_{\mathrm{p}} = J, \qquad \mu = -2h.}$$

This proves an operator identity, not just an agreement of spectra. In the course we take $J > 0$, $h \geq 0$, corresponding to the $\mu \leq 0$ half of the superconducting phase diagram.

The circle now describes the Ising transition

The parameter substitution gives

$$q(k) = 2(h - Je^{ik}), \qquad \boxed{\epsilon(k) = 2\sqrt{J^2 + h^2 - 2Jh\cos k}.}$$

The circle encloses zero when $h < J$ and crosses zero when $h = J$. The superconducting calculation has therefore solved the bulk Ising spectrum and located the exact critical point.

Boundary conditions. The proof above uses an open chain. For a periodic spin chain, the closing Jordan–Wigner string ties the fermionic boundary condition to the spin-parity sector. The infinite-chain dispersion is unaffected by this finite-ring bookkeeping. We return to it in Lecture 4.

The superconducting model was introduced independently. The bond calculation shows why its exact solution also solves the Ising model.

3.7 Recovering spin flips and domain walls

The two limits of Lecture 2 follow by expanding the same square root:

$$h \gg J: \quad \epsilon(k) = 2h\sqrt{1 - 2(J/h)\cos k + (J/h)^2} = 2h - 2J\cos k + O(J^2/h),$$
$$J \gg h: \quad \epsilon(k) = 2J\sqrt{1 - 2(h/J)\cos k + (h/J)^2} = 2J - 2h\cos k + O(h^2/J).$$

	$h \gg J$	$J \gg h$
Spin picture	Reversed X spin	Domain wall
Creation energy in the limit	$2h$	$2J$
Hopping amplitude	$-J$	$-h$
Superconducting bulk phase	Trivial	Topological

The exact fermionic quasiparticles interpolate between these limits. The spin flip and the domain wall remain useful physical pictures of excitations above different backgrounds.

For $J, h \geq 0$, the minimum is at $k = 0$:

$$\boxed{\Delta_{\text{bulk}} = 2|J - h|.}$$

At $h = J$, $\epsilon(k) = 4J|\sin(k/2)| \simeq 2J|k|$ for small k. Near the transition,

$$\epsilon(k) \simeq 2\sqrt{(h - J)^2 + Jh\,k^2}.$$

This is a bulk single-quasiparticle gap. A specified parity sector may require two excitations. The exponentially small splitting of an open-chain doublet is a separate energy scale.

3.8 What does the mapping mean physically?

The open spin and fermion Hamiltonians have the same spectrum, but their physical notions of locality and allowed perturbations differ. For example,

$$Z_1 = c_1 + c_1^\dagger.$$

A longitudinal field $-gZ_1$ is an allowed local spin perturbation. The corresponding fermion term is parity odd and is forbidden in the Hamiltonian of an isolated parity-conserving fermion system. Thus Ising magnetic order and fermionic topological superconductivity are not interchangeable physical classifications. The mapping does not give the Ising chain intrinsic topological order of the toric-code kind.

The question for Lecture 4: When the bulk circle encloses zero, what appears at an open end? We will introduce Majorana operators, construct the end modes, and relate their overlap to the tunneling of Lecture 2.

Reading: Kitaev, *Unpaired Majorana fermions in quantum wires*, Secs. 1–2; Kitaev–Laumann, *Topological phases and quantum computation*, Sec. 2.2; Balents, *Notes for Physics 217A*, quantum Ising sections. In KL, $(\sigma^x, \sigma^z, h_z)$ correspond to (Z, X, h) here.

3.9 Outlook: two extensions of symmetry

Recall the global spin flip of the Ising model:

$$Q = \prod_{j=1}^{N} X_j, \qquad QHQ = H, \qquad Q^2 = I.$$

Two properties can be read off immediately:

$$\underbrace{QZ_jQ = -Z_j}_{\text{a local operator carries charge}} \quad , \qquad \underbrace{Q^{-1} = Q}_{\text{the symmetry is invertible}} \quad .$$

Here "charged" means that the operator transforms nontrivially under the symmetry; it need not carry electric charge. These two properties suggest two independent directions in which to broaden the notion of symmetry.

Higher-form symmetry: from charged points to charged lines

The Ising spin flip is an ordinary, or *0-form*, symmetry: Z_j is a charged local (point) operator. A *1-form* symmetry instead has charged line operators. A lattice example can take the form

$$\mathcal{W}(C) = \prod_{e \in C} O_e,$$

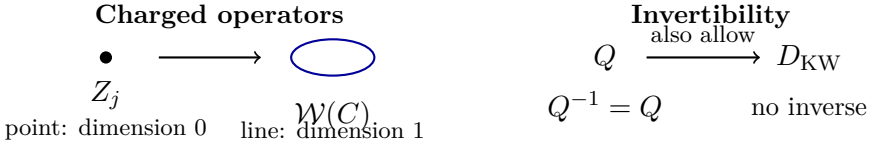
where C is a path or loop and O_e acts on a link. Wilson loops in gauge theories provide examples; we will meet this viewpoint with the toric code. The form degree refers to the dimension of the *charged operator*. The product form of Q, or the presence of the Jordan–Wigner string W_j, does not by itself imply a higher-form symmetry.

Non-invertible symmetry: allowing actions without an inverse

Kramers–Wannier duality exchanges Ising bond and transverse-field terms, and hence J and h. For a *periodic* spin chain, accounting for the boundary conditions and symmetry sectors leads to a duality operator D_{KW}. At the self-dual point,

$$\boxed{J = h: \qquad [D_{\mathrm{KW}}, H] = 0, \qquad D_{\mathrm{KW}}^{-1} \text{ does not exist on the full spin Hilbert space.}}$$

This is an example of a non-invertible symmetry. The ordinary spin flip Q remains a symmetry as well. Lecture 7 will construct D_{KW} and explain how the sector projection produces its kernel.



Two independent extensions: higher-form symmetry concerns the dimension of charged operators; non-invertible symmetry concerns whether a symmetry action has an inverse. One extension does not require the other.

Further reading: Gaiotto et al., *Generalized Global Symmetries*; Shao, *TASI Lectures on Non-Invertible Symmetries*; Seiberg, Seifnashri and Shao, *Non-invertible symmetries and LSM-type constraints*.

Supplement A: a direct two-state calculation

An optional Fock-space check of the BdG result and the paired vacuum.

Four physical states for one momentum pair

For $0 < k < \pi$, the contribution of the distinct pair $(k, -k)$ is

$$H_{k,-k} = \xi_k(n_k + n_{-k} - 1) - i\delta_k(c_k c_{-k} + c_k^\dagger c_{-k}^\dagger).$$

Use $|00\rangle = |\text{vac}\rangle$, $|10\rangle = c_k^\dagger|00\rangle$, $|01\rangle = c_{-k}^\dagger|00\rangle$, and $|11\rangle = c_k^\dagger c_{-k}^\dagger|00\rangle$. The odd states $|10\rangle, |01\rangle$ have energy zero. Pairing mixes only $|00\rangle$ and $|11\rangle$. In particular, $c_k c_{-k}|11\rangle = -|00\rangle$. Thus, in the ordered even basis $(|00\rangle, |11\rangle)$,

$$H_{k,-k}^{\text{even}} = \begin{pmatrix} -\xi_k & i\delta_k \\ -i\delta_k & \xi_k \end{pmatrix}, \qquad \det(H_{k,-k}^{\text{even}} - \lambda I) = \lambda^2 - \xi_k^2 - \delta_k^2.$$

The four energies are $-E, 0, 0, +E$. Relative to the ground state, they are $0, E, E, 2E$: two independent quasiparticles, each costing $E(k)$. The gap within the even block is $2E(k)$.

The even Fock-space matrix is $H_{k,-k}^{\text{even}} = -\mathcal{H}(k)$ in our conventions. These matrices act on different objects: coefficients of $|00\rangle, |11\rangle$ versus particle–hole amplitudes. Do not identify a Nambu column of operators with a two-component many-body state.

Coherence factors and the relative phase

For $\Delta_p > 0$, $\delta_k > 0$ on $0 < k < \pi$. The even ground state is

$$|\Omega_k\rangle = u_k|00\rangle + v_k|11\rangle, \qquad u_k = \sqrt{\frac{E(k) + \xi_k}{2E(k)}}, \qquad v_k = i\sqrt{\frac{E(k) - \xi_k}{2E(k)}}.$$

Substitution gives $H_{k,-k}^{\text{even}}(u_k, v_k)^T = -E(k)(u_k, v_k)^T$. For $\xi_k = 0$ and $\delta_k > 0$, the check reduces to

$$\begin{pmatrix} 0 & i\delta_k \\ -i\delta_k & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = -\delta_k \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Thus the relative phase is $+i$ in our convention. If $\xi_k \gg |\delta_k|$ the pair is mostly empty; if $\xi_k \ll -|\delta_k|$ it is mostly occupied.

A vacuum of quasiparticles

Choose the canonical annihilation operators

$$\eta_k = u_k c_k - v_k c_{-k}^\dagger, \qquad \eta_{-k} = u_k c_{-k} + v_k c_k^\dagger.$$

For example, $c_k|\Omega_k\rangle = v_k|01\rangle$ and $c_{-k}^\dagger|\Omega_k\rangle = u_k|01\rangle$, so $\eta_k|\Omega_k\rangle = 0$; similarly $\eta_{-k}|\Omega_k\rangle = 0$. Direct substitution also gives

$$H_{k,-k} = E(k)(\eta_k^\dagger \eta_k + \eta_{-k}^\dagger \eta_{-k} - 1).$$

Supplement B: the BCS state and the bulk invariant

Optional detail on the many-body wavefunction and the circle criterion.

Assembling the BCS state

Using the coefficients in Supplement A, the ground state for paired momenta is

$$|\Omega\rangle = \prod_{0<k<\pi} \left(u_k + v_k c_k^\dagger c_{-k}^\dagger\right) |\mathrm{vac}\rangle,$$

with any self-conjugate momenta handled separately. At $k = 0, \pi$, the Hamiltonian is $\xi_k(n_k - 1/2)$, so the ground occupation is 0 for $\xi_k > 0$ and 1 for $\xi_k < 0$. At $\xi_k = 0$ the bulk gap closes. A fixed total-parity constraint must also be imposed when appropriate.

The product is over momentum pairs, not over real-space sites. Thus solving the Hamiltonian as independent quasiparticles does not imply that its ground state is a product state of the original local degrees of freedom. Entanglement between k and $-k$ should not be confused with entanglement between spatial regions, which we will study later.

A Bogoliubov vacuum is empty of quasiparticles. It generally contains a coherent superposition of different physical pair occupations. The next spin example, AKLT, will likewise be understood through an explicit wavefunction.

Why the direction of q matters

Since $q = \xi - i\delta$, write $q = Ee^{i\theta}$. Then

$$\frac{\mathcal{H}(k)}{E(k)} = \cos\theta(k)\,\tau_z - \sin\theta(k)\,\tau_y.$$

The angle determines the particle–hole eigenvector, while E fixes its energy. For a fixed physical unit cell and real uniform parameters, the integer winding of q is well defined whenever the spectrum is gapped.

The parity of this winding is the Kitaev-chain $\mathbb{Z}_2$ invariant. For $w > 0$ and real $\Delta_\mathrm{p} \neq 0$,

$$\boxed{\mathcal{M} = \mathrm{sgn}[\xi_0 \xi_\pi] = \mathrm{sgn}[(\mu + 2w)(\mu - 2w)].}$$

Thus $\mathcal{M} = -1$ for $|\mu| < 2w$. With negative real pairing the winding orientation reverses, but its parity and $\mathcal{M}$ are unchanged. The integer winding uses the extra real-parameter structure of this model; the robust distinction of a general parity-conserving superconducting wire is the $\mathbb{Z}_2$ one.

If $\Delta_\mathrm{p} = 0$ and $|\mu| < 2w$, the bulk is gapless: the curve collapses onto a segment through zero. A gapped-phase invariant cannot be assigned there by this argument. The formula must always be used with its gap condition.