Statistical Physics

Introduction to Topological Order

Takashi Oka

4 Majorana end modes and finite-size splitting

Lecture 3 distinguished two bulk phases using $q(k) = 2(h - Je^{ik})$. We now open the chain, construct its end modes, and connect their overlap to tunneling and quantum information. We use $J > 0$, $h \geq 0$, with $w = \Delta_{\mathrm{p}} = J$ and $\mu = -2h$.

4.1 From complex fermions to Majorana operators

Resolve each fermion into two Hermitian operators:

$$\boxed{a_j = c_j + c_j^\dagger, \qquad b_j = \frac{c_j - c_j^\dagger}{i}, \qquad c_j = \frac{a_j + ib_j}{2}.}$$

The fermionic algebra immediately gives

$$a_j^2 = b_j^2 = I, \quad \{a_j, a_k\} = \{b_j, b_k\} = 2\delta_{jk}, \quad \{a_j, b_k\} = 0.$$

For example, $(c_j + c_j^\dagger)^2 = c_j c_j^\dagger + c_j^\dagger c_j = I$. Also $ia_jb_j = 2n_j - 1$, so occupation and parity become

$$\boxed{n_j = \frac{1 + ia_jb_j}{2}, \qquad Q = \prod_j (-ia_jb_j) = (-1)^{\sum_j n_j}.}$$

A single Majorana has no separate empty/occupied states: $a_j^\dagger a_j = I$. A pair defines one fermion occupation. The chain still has N fermion modes and 2^N states.

The bond identity of Lecture 3 and the on-site term give

$$J(c_j - c_j^\dagger)(c_{j+1} + c_{j+1}^\dagger) = iJb_ja_{j+1}, \qquad 2h(n_j - \tfrac{1}{2}) = iha_jb_j.$$

Thus the Hamiltonian is a chain of Majorana couplings:

$$\boxed{H = ih \sum_{j=1}^{N} a_jb_j + iJ \sum_{j=1}^{N-1} b_ja_{j+1}.}$$

An isolated pair already shows the energy scale. For $f = (\gamma_1 + i\gamma_2)/2$,

$$i\lambda\gamma_1\gamma_2 = \lambda(2f^\dagger f - 1).$$

For $\lambda > 0$, its energies are $-\lambda, +\lambda$, with excitation cost 2λ. At zero coupling the two occupations are degenerate.

The spin expressions follow from $S_j = \prod_{\ell<j} X_\ell$:

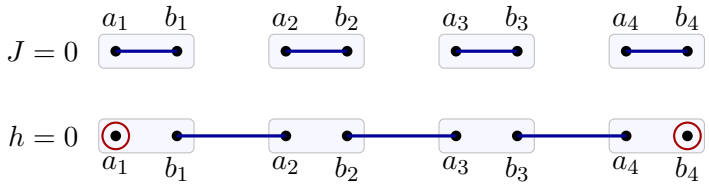
$$a_j = S_jZ_j, \qquad b_j = -S_jY_j, \qquad ib_ja_{j+1} = -Z_jZ_{j+1}.$$

In particular $a_1 = Z_1$, an identity we will use when discussing protection.

Majorana operators expose a pairing pattern in real space. We can now ask whether every operator is coupled, or whether an end remains unpaired.

4.2 Two ways to pair the Majoranas

At $J = 0$, the pairs are on sites: (a_j, b_j). At $h = 0$, they lie on bonds: (b_j, a_{j+1}). Each shaded box is one physical site. Solid lines denote couplings; circles mark operators absent from the Hamiltonian.



At $J = 0$, occupying $c_j = (a_j + ib_j)/2$ costs $2h$: it is the spin flip of Lecture 2. At $h = 0$, define a bond fermion

$$d_j = \frac{b_j + ia_{j+1}}{2}, \qquad \boxed{d_j^\dagger d_j = \frac{1 + ib_j a_{j+1}}{2} = \frac{1 - Z_j Z_{j+1}}{2}.}$$

An occupied bond is a domain wall and costs $2J$. Consequently,

$$H_{h=0} = -J(N-1) + 2J \sum_{j=1}^{N-1} d_j^\dagger d_j.$$

The $N - 1$ bond modes use $2N - 2$ Majoranas. The remaining a_1, b_N form

$$f_{\text{end}} = \frac{a_1 + ib_N}{2}, \qquad [H_{h=0}, a_1] = [H_{h=0}, b_N] = 0.$$

Both occupations of f_{end} have the same energy. This accounts for the two ground states of the open Ising chain. There are still only N fermionic modes, and hence 2^N states in total.

A domain wall is an occupied, coupled bond mode with energy $2J$. An unpaired end Majorana is a different object. Two ends together define one zero-energy occupation at $h = 0$.

4.3 Constructing an end mode at nonzero field

For two Majoranas, $[i\lambda\gamma_1\gamma_2, \gamma_1] = -2i\lambda\gamma_2$ and $[i\lambda\gamma_1\gamma_2, \gamma_2] = 2i\lambda\gamma_1$. Applying these identities to the open chain gives, with $b_0 = a_{N+1} = 0$,

$$[H, a_j] = 2i(Jb_{j-1} - hb_j), \qquad [H, b_j] = 2i(ha_j - Ja_{j+1}).$$

Start with a_1. Its commutator is $-2ihb_1$. Adding $(h/J)a_2$ cancels the b_1 term but leaves a term proportional to b_2. Repeating the cancellation suggests $a_1 + (h/J)a_2 + (h/J)^2 a_3 + \cdots$.

Seek a left-end operator $\Gamma_L = \sum_{j=1}^{N} u_j a_j$. Collecting terms,

$$[H, \Gamma_L] = 2i \sum_{j=1}^{N-1} (Ju_{j+1} - hu_j) b_j - 2ihu_N b_N.$$

Every interior coefficient vanishes if

$$u_{j+1} = \frac{h}{J} u_j.$$

Let $r = h/J < 1$. Normalizing so that $\Gamma_L^2 = I$ gives

$$\boxed{\Gamma_L = \mathcal{N} \sum_{j=1}^{N} r^{j-1} a_j, \qquad \mathcal{N} = \left(\frac{1 - r^2}{1 - r^{2N}} \right)^{1/2}.}$$

The normalization follows from anticommutation: $(\sum_j u_j a_j)^2 = \sum_j u_j^2 I$. Only the far-end term remains:

$$\boxed{[H, \Gamma_L] = -2iJ\mathcal{N} r^N b_N.}$$

Likewise,

$$\Gamma_R = \mathcal{N} \sum_{j=1}^{N} r^{N-j} b_j, \qquad [H, \Gamma_R] = 2iJ\mathcal{N} r^N a_1.$$

These Hermitian operators square to one, anticommute with each other, and anticommute with total parity Q.

Localization and the meaning of "zero mode"

The amplitudes decay as $r^{j-1} = e^{-(j-1)/\xi_{\text{end}}}$, where

$$\boxed{\xi_{\text{end}} = \frac{1}{\ln(J/h)} \qquad (0 < h < J).}$$

A semi-infinite chain has an exactly normalizable zero mode for $h < J$. A finite chain generally has only an *approximate* zero mode: the small commutator above measures the effect of the opposite end. At $h = 0$ the modes are exactly a_1, b_N even for finite N.

As $h \to J^-$, the localization length diverges. For $h \geq J$, this recurrence gives no normalizable end mode on a semi-infinite uniform chain.

A normalizable end mode exists precisely when $h < J$, the condition for Lecture 3's circle to enclose zero. This is the bulk–boundary correspondence in this chain. Its tail reaches the other end and produces finite-size splitting.

4.4 Overlap of the two ends

To leading order for a long chain, the low-energy Hamiltonian is

$$H_{\text{low}} = E_{\text{c}} + it_N\Gamma_L\Gamma_R = E_{\text{c}} + t_N(2f_{\text{end}}^\dagger f_{\text{end}} - 1), \qquad f_{\text{end}} = \frac{\Gamma_L + i\Gamma_R}{2}.$$

Here E_{c} is the center of the ground-state doublet. To estimate t_N, write $H = ia^T Kb$ with $K_{jj} = h$, $K_{j+1,j} = -J$. The left and right coefficient vectors are $u_j = \mathcal{N}r^{j-1}$ and $v_j = \mathcal{N}r^{N-j}$. The cancellation gives $u^T K = (0, \ldots, 0, J\mathcal{N}r^N)$, so

$$t_N \simeq u^T K v = J\mathcal{N}^2 r^N.$$

This projection gives the leading coupling, not an exact finite-N diagonalization. For fixed $r < 1$ and $N \gg \xi_{\text{end}}$,

$$\boxed{\Delta E_N \simeq 2J(1-r^2)r^N \;\propto\; e^{-N/\xi_{\text{end}}}.}$$

The gap between the two occupations is $2t_N$, not t_N.

4.5 The same splitting from high-order perturbation theory

In the spin picture, let

$$|U\rangle = |0\cdots 0\rangle, \quad |D\rangle = |1\cdots 1\rangle, \qquad V = -h\sum_j X_j.$$

A single X_j flips one spin. Connecting $|U\rangle$ to $|D\rangle$ therefore requires at least N applications of V. One path is

$$|0000\rangle \xrightarrow{X_1} |1000\rangle \xrightarrow{X_2} |1100\rangle \xrightarrow{X_3} |1110\rangle \xrightarrow{X_4} |1111\rangle.$$

A wall enters at one end, crosses the chain, and leaves at the other. Summing all leading-order paths gives, for fixed $N \geq 2$ and $h/J \ll 1$,

$$H_{\text{eff}} = (E_0 + \delta E)I - t_N\overline{X}, \qquad t_N = \frac{h^N}{J^{N-1}}[1 + O((h/J)^2)].$$

Here $\overline{X} = |U\rangle\langle D| + |D\rangle\langle U|$. Thus $\Delta E_N = 2t_N \simeq 2J(h/J)^N$, in agreement with the weak-field limit of the end-mode result. Supplement A explains the prefactor and a three-spin check. The finite-h low-energy states are dressed versions of the Ising cat states.

Two different gaps: the bulk quasiparticle gap approaches $2(J-h)$ in the ordered phase; the open-chain doublet splitting is exponentially small in N. A fixed-parity sector excludes one member of this doublet.

4.6 What changes when the chain is closed?

Closing a physical fermion chain

At $h = 0$, the open chain leaves a_1, b_N unpaired. A closing fermionic bond

$$H_{\text{close}} = iJb_N a_1 = -iJa_1 b_N$$

couples them directly. Since $ia_1 b_N = 2n_{\text{end}} - 1$,

$$H_{\text{close}} = J(1 - 2n_{\text{end}}).$$

The two end occupations now differ in energy by $2J$, rather than an exponentially small amount. All Majoranas are paired and the full periodic fermionic chain has a unique ground state at this limit. Its ground-state parity is selected by the boundary condition. Restricting to the other parity sector forces an excitation.

Closing the chain changes its boundary, not its bulk parameters. At $h = 0$, Lecture 3's curve remains $q(k) = -2Je^{ik}$: it winds once around zero, and the bulk gap is still $2J$. A unique ground state on the fermionic ring is therefore compatible with a topological bulk. Cutting the bond exposes the end modes again. An interface with a trivial region can serve as an end too.

The end degeneracy is a boundary consequence of the bulk phase. Removing the boundary does not remove the topological phase.

A periodic spin chain has a different closing term

It is tempting to say that closing the Ising chain also removes its magnetic doublet. It does not: at $h = 0$, both $|U\rangle$ and $|D\rangle$ remain ground states of the periodic ferromagnet. The Jordan–Wigner boundary term resolves this. From $Q = \prod_j X_j$ and $b_N = -S_N Y_N$,

$$ib_N a_1 = QZ_N Z_1, \qquad \boxed{-JZ_N Z_1 = -iJQ\, b_N a_1.}$$

Thus the fermionic closing bond depends on the spin-parity sector $Q = q$. Equivalently, the fermions obey $c_{N+1} = -q\, c_1$: even spin parity gives an antiperiodic fermion boundary condition, odd parity a periodic one.

In the $h = 0$ open-chain ground space,

$$Q = -ia_1 b_N = 1 - 2n_{\text{end}},$$

so the spin closing term is simply $-JQ^2 = -J$. It shifts both ground states equally. This calculation explains why the two ways of closing the chain are physically different even though the open-chain spectra coincide.

Open spin and fermion chains are exactly related. On a ring, the Jordan–Wigner string ties the boundary condition to parity. One must specify which physical chain and which sector are being closed.

4.7 Why can the boundary degeneracy be robust?

We follow the locality-and-symmetry reasoning of KL Sec. 2.5. Keep the bulk in its ground state and first neglect the exponentially small end overlap.

The algebra in the low-energy space

The end operators obey

$$\Gamma_L^2 = \Gamma_R^2 = I, \quad \{\Gamma_L, \Gamma_R\} = 0, \quad \{Q, \Gamma_L\} = \{Q, \Gamma_R\} = 0.$$

If $[H, \Gamma_L] = 0$, applying Γ_L changes parity without changing energy: it gives an orthogonal partner. For a long finite chain this pairing is approximate. A general Hermitian perturbation on the two-state end space is

$$V_{\text{low}} = v_0 I + v_L \Gamma_L + v_R \Gamma_R + v_{LR} i \Gamma_L \Gamma_R.$$

Parity forces $v_L = v_R = 0$. Locality suppresses v_{LR} for widely separated ends in a gapped bulk: a local perturbation couples them only through small tails or virtual propagation. An even polynomial of one isolated Majorana is a scalar. Directly coupling both ends can nevertheless split the doublet.

Why the spin realization is more vulnerable

At $h = 0$, $a_1 = Z_1$ and

$$|C_\pm\rangle = \frac{|U\rangle \pm |D\rangle}{\sqrt{2}}, \qquad Z_1 |C_\pm\rangle = |C_\mp\rangle.$$

The allowed spin perturbation $-gZ_1$ splits $|U\rangle, |D\rangle$ by $2|g|$: the local order parameter is a logical phase error. The fermion term $-ga_1$ is parity odd and cannot occur in an isolated parity-conserving Hamiltonian. Exchange of a fermion with a reservoir can change the wire's parity.

What survives when interactions are added?

For a concrete local interaction, consider

$$H(g) = H + g \sum_{j=1}^{N-1} (n_j - \tfrac{1}{2})(n_{j+1} - \tfrac{1}{2}), \qquad [H(g), Q] = 0.$$

The extra term contains quartic fermion products, so the quadratic solution no longer applies. Even at $h = 0$, the bare operator a_1 generally fails to commute with $H(g)$. This does not by itself imply that the boundary degree of freedom has disappeared.

Start in $h < J$ and turn on a weak g continuously, preserving parity and the bulk gap. The low-energy states change, and their end operators must change with them: $\Gamma_L \to \widetilde{\Gamma}_L(g)$. This *dressed* operator can contain products of several fermion operators. It remains concentrated near the end, with decaying tails. Within the low-energy space it still changes parity and connects the two nearly degenerate states. The parity-and-locality argument above then continues to apply.

This is a gapped-phase argument, not a new exact solution. It does not claim an operator commuting with the full interacting Hamiltonian on every excited state. The controlled construction is called *quasi-adiabatic continuation*; see Hastings–Wen (2005), as cited by KL. No proof of that construction is needed here.

Interactions change the wavefunctions and the end operators. With a local, parity-preserving deformation that keeps the bulk gap open, the low-energy boundary structure survives.

4.8 Four Majoranas and a logical qubit

Two Majoranas give two occupations only when both parities are included. If total parity and the bulk occupations are fixed, a single pair leaves just one state. To encode a qubit in one parity sector, use four separated modes:

$$f_1 = \frac{\gamma_1 + i\gamma_2}{2}, \qquad f_2 = \frac{\gamma_3 + i\gamma_4}{2}, \qquad \mathcal{P} = (1-2n_1)(1-2n_2).$$

Choose even parity for these modes. The allowed states are

$$\boxed{|0_L\rangle = |00\rangle, \qquad |1_L\rangle = |11\rangle = f_1^\dagger f_2^\dagger |00\rangle.}$$

The creation-operator order fixes the sign convention. A coherent logical state $\alpha|0_L\rangle + \beta|1_L\rangle$ now stays within one total-parity sector. For $2m$ Majorana modes and fixed bulk occupations, fixing total parity leaves 2^{m-1} states rather than 2^m.

Logical operations are even Majorana products

In this basis,

$$\boxed{\overline{Z} = -i\gamma_1\gamma_2 = 1-2n_1, \qquad \overline{X} = -i\gamma_2\gamma_3.}$$

They square to I, preserve total parity, and anticommute. Their actions are

$$\overline{Z}|00\rangle = |00\rangle, \quad \overline{Z}|11\rangle = -|11\rangle, \qquad \overline{X}|00\rangle = |11\rangle, \quad \overline{X}|11\rangle = |00\rangle.$$

Thus a coupling $i\varepsilon\gamma_1\gamma_2 = -\varepsilon\overline{Z}$ can produce an intended logical rotation, or an unwanted logical error. Fermion parity does not distinguish the two logical states and does not forbid this coupling. Spatial separation controls unwanted overlap; operations require controlled access to the encoded information.

4.9 What we have learned, and the move to AKLT

The Ising spin doublet is distinguished by local magnetization. With fermionic locality and parity imposed, the $h < J$ wire is a topological superconducting phase with end modes. These are different physical interpretations of the same open-chain spectrum. Neither the mapping nor an edge degeneracy alone establishes intrinsic topological order of the two-dimensional toric-code kind.

The bulk winding of Lecture 3 predicts the end modes constructed here. Their overlap links three viewpoints: a domain wall tunneling across a spin chain, a small coupling between end Majoranas, and a logical operation on a low-energy space. In the next two lectures we return to spins and build the AKLT wavefunction, asking how boundary spin-1/2 degrees of freedom can exist without bulk magnetic order and how symmetry protects them.

Reading: Kitaev–Laumann, *Topological phases and quantum computation*, Secs. 2.1 and 2.3–2.5; Kitaev, *Unpaired Majorana fermions in quantum wires*. The fixed-parity encoding and fusion outlook connect to Nayak et al., *Non-Abelian Anyons and Topological Quantum Computation*.

Supplement A: the leading tunneling matrix element

Optional detail for the connection to Lecture 2.

A resolvent expression

Let $H_0 = -J\sum_{j=1}^{N-1} Z_j Z_{j+1}$, $E_0 = -J(N-1)$ and

$$P = |U\rangle\langle U| + |D\rangle\langle D|, \qquad R = \sum_{m\notin\mathcal{C}} \frac{|m\rangle\langle m|}{E_0 - E_m}, \quad \mathcal{C} = \mathrm{span}\{|U\rangle, |D\rangle\}.$$

For $V = -h\sum_j X_j$, the first nonzero off-diagonal contribution is

$$\boxed{A_N = \langle D|V(RV)^{N-1}|U\rangle.}$$

This is the leading off-diagonal term, not a formula for the entire effective Hamiltonian at all orders. All intermediate states lie outside $\mathcal{C}$. Every spin must flip exactly once at order N.

For a left-to-right path there is one wall, costing $2J$, at every intermediate step. That single path contributes

$$A_{\mathrm{path}} = \frac{(-h)^N}{(-2J)^{N-1}} = -\frac{h^N}{(2J)^{N-1}}.$$

It is only one of $N!$ flip orderings. Other paths contain more walls and different denominators, so multiplying this amplitude by $N!$ is incorrect.

Three spins: sum the six paths

For $N = 3$, two orderings, $(1,2,3)$ and $(3,2,1)$, have intermediate costs $(2J, 2J)$. The other four have costs $(2J, 4J)$ or $(4J, 2J)$. Therefore

$$A_3 = -h^3\left[\frac{2}{(2J)^2} + \frac{4}{(2J)(4J)}\right] = -\frac{h^3}{J^2}, \qquad \Delta E_3 = 2|A_3| + O(h^5/J^4).$$

All paths contribute with the same negative sign for $J, h > 0$.

Check the leading coefficient for general N

The finite-chain matrix K in Supplement D has $\det K = h^N$. Its singular values obey $\prod_\nu s_\nu = |\det K|$. As $h/J \to 0$, one is small and the other $N-1$ approach J, so

$$s_{\min} = \frac{h^N}{\prod_{\nu\neq\min} s_\nu} \sim \frac{h^N}{J^{N-1}}, \qquad \Delta E_N = 2s_{\min} \sim \frac{2h^N}{J^{N-1}}.$$

This fixes the coefficient obtained by summing all leading perturbative paths. It also separates two limits: the perturbative result holds for fixed N and small h/J; the overlap formula in Sec. 4.4 is a long-chain asymptotic at fixed $h/J < 1$. They agree in their common regime.

Supplement B: pair occupation and fusion channels

A short outlook. This is not needed for the end-mode derivation.

Couple two separated zero modes

Bring two Majorana-carrying defects together, so that their modes acquire a coupling. Choose the convention

$$f = \frac{\gamma_1 + i\gamma_2}{2}, \qquad H_{\text{pair}} = i\lambda\gamma_1\gamma_2 = \lambda(2n-1), \quad \lambda > 0.$$

The previously almost degenerate occupations now have different energies:

Occupation	Pair parity $1-2n$	Energy	Channel
$n=0$	$+1$	$-\lambda$	1 (vacuum)
$n=1$	-1	$+\lambda$	ψ (fermion)

The occupied channel leaves a fermionic excitation relative to the empty reference state. In the Ising anyon theory these two possibilities are written

$$\boxed{\sigma \times \sigma = 1 + \psi.}$$

The symbol σ labels an anyon or a suitable defect carrying a Majorana mode; it is not the operator γ or an ordinary Ising up/down domain wall. The + lists allowed channels. It does not specify a superposition or assign probability 1/2 to each outcome.

The related charge rules are

$$\psi \times \psi = 1, \qquad \sigma \times \psi = \sigma.$$

Two fermion charges have total vacuum charge. This does not mean two particles automatically disappear or have zero energy. Absorbing a fermion at a Majorana-carrying defect leaves its type unchanged; with a distant partner, the pair occupation can change while total parity is conserved.

The fixed-parity qubit in this language

For four modes paired as $(1,2)$ and $(3,4)$, the even sector has

State	First pair	Second pair	Total channel
$\lvert 00\rangle$	1	1	1
$\lvert 11\rangle$	ψ	ψ	1

Measuring the first pair's parity measures $\overline{Z}$. Its projectors are $\Pi_1 = (I+\overline{Z})/2$ and $\Pi_\psi = (I-\overline{Z})/2$; the probabilities on $\alpha|0_L\rangle + \beta|1_L\rangle$ are $|\alpha|^2$ and $|\beta|^2$. This reads logical information, unlike the repetition-code syndrome checks, which have the same values on both logical basis states.

Scope. Wire ends illustrate the occupation structure underlying Ising fusion. A one-dimensional chain alone does not provide exchangeable two-dimensional anyons or derive their braiding statistics. A magnetic domain wall and a topological/trivial superconducting interface must not be confused.

Further reading: C. Nayak et al., *Non-Abelian Anyons and Topological Quantum Computation*, Secs. II.A and IV.A; D. A. Ivanov, *Non-abelian statistics of half-quantum vortices in p-wave superconductors.*

Supplement C: general quadratic Majorana Hamiltonians

Optional linear algebra behind the independent-quasiparticle description.

Write $\gamma = (a_1, b_1, a_2, b_2, \ldots)^T$ and

$$H = \frac{i}{4} \sum_{\alpha,\beta=1}^{2N} A_{\alpha\beta}\gamma_\alpha\gamma_\beta, \qquad A^T = -A, \quad A \text{ real.}$$

For the chain, $A_{2j-1,2j} = 2h$ and $A_{2j,2j+1} = 2J$, with opposite signs on reversing the indices. Both orders occur in the sum, explaining $1/4$ and the factors of two in A.

A change of basis makes independent pairs

A real orthogonal matrix O gives the canonical form

$$O^T A O = \bigoplus_{\nu=1}^{N} \begin{pmatrix} 0 & \epsilon_\nu \\ -\epsilon_\nu & 0 \end{pmatrix}, \qquad \epsilon_\nu \geq 0.$$

Because $O^T O = I$, the new operators $\widetilde{\gamma} = O^T\gamma$ obey the same anticommutation relations. With $\eta_\nu = (\widetilde{\gamma}_{2\nu-1} + i\widetilde{\gamma}_{2\nu})/2$,

$$\boxed{H = \sum_{\nu=1}^{N} \epsilon_\nu(\eta_\nu^\dagger\eta_\nu - \tfrac{1}{2}).}$$

The eigenvalues of A are $\pm i\epsilon_\nu$. A $2N \times 2N$ matrix thus determines all 2^N many-body energies. Zero ϵ_ν means that the corresponding occupation is unconstrained by energy, subject to any imposed parity constraint.

What does "free" mean?

For positive ϵ_ν, the ground state satisfies $\eta_\nu|\Omega\rangle = 0$. In terms of the original orbitals,

$$\eta_\nu = \sum_j (u_{\nu j}c_j + v_{\nu j}c_j^\dagger).$$

This is a Bogoliubov transformation. The quasiparticle vacuum is generally a coherent superposition of physical pair occupations, as calculated in Lecture 3, Supplement A. Free quasiparticles mean additive occupation energies, not an unentangled state of the original sites.

The orthogonal transformation used to find a canonical form is a mathematical basis change. It need not be a physically allowed parity-preserving time evolution. In particular, imposing a physical parity sector remains a separate step after diagonalization.

The exact solution uses independent coupled pairs after a basis change. The special role of an end mode is that the required pair is spatially separated, so its coupling can be exponentially small.

Supplement D: the exact finite open chain

Optional linear algebra. No new physical assumption is needed.

Replace the many-body matrix by an $N \times N$ matrix

Write $a = (a_1, \ldots, a_N)^T$, $b = (b_1, \ldots, b_N)^T$ and

$$H = i a^T K b, \qquad K = \begin{pmatrix} h & 0 & \cdots & 0 \\ -J & h & \ddots & \vdots \\ 0 & -J & \ddots & 0 \\ \vdots & \ddots & -J & h \end{pmatrix}.$$

Take a real singular-value decomposition

$$K = O_a \, \mathrm{diag}(s_1, \ldots, s_N) \, O_b^T, \qquad s_\nu \geq 0.$$

The matrices O_a, O_b are orthogonal. Therefore $\widetilde{a} = O_a^T a$, $\widetilde{b} = O_b^T b$ obey the same Majorana algebra. With $\eta_\nu = (\widetilde{a}_\nu + i\widetilde{b}_\nu)/2$,

$$\boxed{H = \sum_{\nu=1}^{N} s_\nu (2\eta_\nu^\dagger \eta_\nu - 1), \qquad \epsilon_\nu = 2s_\nu.}$$

This is exact for finite N. The ground energy is $-\sum_\nu s_\nu$; all other energies follow by filling modes. No parity restriction is needed to recover the full spectrum of the open spin chain. A physical fermionic system with fixed total parity uses only the appropriate subset of occupations.

A two-spin check

For $N = 2$,

$$K = \begin{pmatrix} h & 0 \\ -J & h \end{pmatrix}, \qquad s_\pm = \frac{\sqrt{J^2 + 4h^2} \pm J}{2}.$$

The four many-body energies are

$$-(s_+ + s_-), \quad -(s_+ - s_-), \quad s_+ - s_-, \quad s_+ + s_-,$$

or $-\sqrt{J^2 + 4h^2}, -J, J, \sqrt{J^2 + 4h^2}$. Thus the lowest splitting is

$$\Delta E_2 = 2s_- = \sqrt{J^2 + 4h^2} - J = \frac{2h^2}{J} + O(h^4/J^3),$$

exactly as in the Bell-basis calculation of Lecture 2.

What happens at the two endpoints?

At $J = 0$, every singular value equals h. At $h = 0$, there are $N - 1$ singular values equal to J and one equal to zero. The zero singular value is the finite-chain signature of the unpaired end Majoranas. The main text constructs those operators explicitly and follows them away from $h = 0$.